Math 601 – **Spring** 2014
Homework #2

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1. (Hatcher, number 1.2.14) Consider the quotient of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group of order eight.

2. (Hatcher, number 1.3.9) Show that if a path-connected, locally path-connected space X has a finite fundamental group, then every map $X \to S^1$ is null-homotopic.

3. (a) (Hatcher, number 1.3.10) Find all connected 2-sheeted and 3-sheeted covers of $S^1 \vee S^1$, up to isomorphism of covering spaces without base points.
   (b) Up to conjugacy, how many ways can $\mathbb{F}_4$ embed into $\mathbb{F}_2$?

4. (Hatcher, number 1.3.14) Find all the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.

5. Let X be the space obtained by identifying the north and south poles of S^2. Describe all coverings spaces of X.

6. (Hatcher, number 1.3.18) For a path-connected, locally path-connected, and semi-locally simply-connected space X, call a path-connected covering space $\widetilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a 'universal abelian cover' is unique up to isomorphism. Describe the covering space explicitly for $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.

7. Let G be a path-connected topological group.
   (a) Show that any path-connected covering space of G is also a topological group.
   (b) Show that a covering space of G that is not path-connected is not necessarily a topological group.

8. Describe all normal covers of the Klein bottle (a cell complex for the Klein bottle is available at its wikipedia page)

9. For $n \geq 2$, show that $\pi_n(X, x_0)$ is abelian.